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Kinematic and mechanical discontinuity at a coherent interface

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Abstract—Across a coherent interface, displacements are continuous, but some components of large strain, rotation, stretching, spin, stress and rheology need not be so. This accounts for refraction of cleavage and other geological structures.

The general theory of deformation and motion is used to analyze how discontinuities develop. A measure of discontinuity in total deformation gradients (strain and rotation) is the ratio, (K), of amounts of shear above and below the interface and in directions parallel to it. It is shown that (K) is equal to the ratio of simple shearings above and below the interface provided (i) the latter ratio is constant in time and (ii) no volume changes occur.

These kinematic conditions are shown to hold in Newtonian fluids and incompressible neo-Hookean solids, where (K) is exactly equal to the inverse viscosity ratio, or to the inverse rigidity ratio. The conditions do not hold in general in Reiner–Rivlin fluids. In power-law fluids, the ratio of simple shearings is constant for two special classes of motion, one with little simple shearing along the interface, the other with simple shearing alone; therefore, the rheological contrast can be determined.

The theoretical results can be used to determine rheological contrasts in nature or experiment, provided there is some knowledge of the nature of the flow laws that operate. Under these conditions, an interface is an inbuilt rheometer.

INTRODUCTION

INTERFACES between rocks of differing rheological properties exist at all scales in the Earth's crust, from that of grains embedded in a matrix, to that of igneous plutons surrounded by country rock. The very heterogeneity of the crust is responsible for the frequent occurrence of interfaces. Good examples occur in any stratified series of sediments, where microstructure and chemical composition are discontinuous at bedding surfaces.

Rocks from tectonically deformed regions frequently show evidence of large ductile strains, with accompanying formation of tectonic cleavage, parallel or nearly parallel to principal directions of finite strain (see discussion by Williams 1977). At interfaces, the state of strain may be discontinuous. Thus, it is a common observation that cleavage refracts at coherent interfaces. The amount of cleavage refraction is potentially a useful measure of rheological contrast: the field geologist uses cleavage refraction in a qualitative way to estimate relative 'competence' of rock units and to identify graded bedding, which he then uses as a way-up criterion in structurally complex terrains.

The purpose of this paper is to explore some quantitative aspects of large ductile strains at interfaces, with a view to determining rheological contrasts where possible. Although geologically it might seem sensible to start by considering refraction of cleavage (or principal directions of large strain), there is not enough information in this alone to enable one to calculate rheological contrasts. Instead the focus here will be on refraction of passive markers (lines or planes) known to have been normal to an interface before deformation. The mathematics of this problem are simpler.

Each rock type is assumed to be a continuous and homogeneous medium, separated from adjacent media by coherent interfaces. Coherence of an interface implies that certain parameters must be continuous across it, although others need not be so. Examples of parameters that must be continuous are (i) displacements (vector components both normal and parallel to the interface), (ii) certain tensor components of stress (shear components parallel to the interface and the normal component across it) and (iii) certain tensor components of large strain, infinitesimal strain or strain-rate (normal components parallel to the interface and shear components across it). Other stress or strain components may be discontinuous and hence the principal directions may refract across the interface.

For Newtonian fluids, Treagus (1973, 1981) has shown that principal stresses refract by an amount that depends upon the viscosity ratio across the interface. A similar analysis is given by Goguel (1982) for ideal plastic materials. This paper will emphasize variation, not only of stresses or small strains, but of large strains. This makes the analysis applicable to a broad range of tectonic situations.

The first part of this paper deals with the kinematics of an interface, that is, with (i) the total deformation and (ii) its time history, the motion. The second part considers mechanical aspects, including stress conditions and various models of rheological behaviour.

KINEMATICS

Much of the following theory and nomenclature is based on the excellent review by Truesdell & Toupin (1960). For the sake of clarity, some of their equations

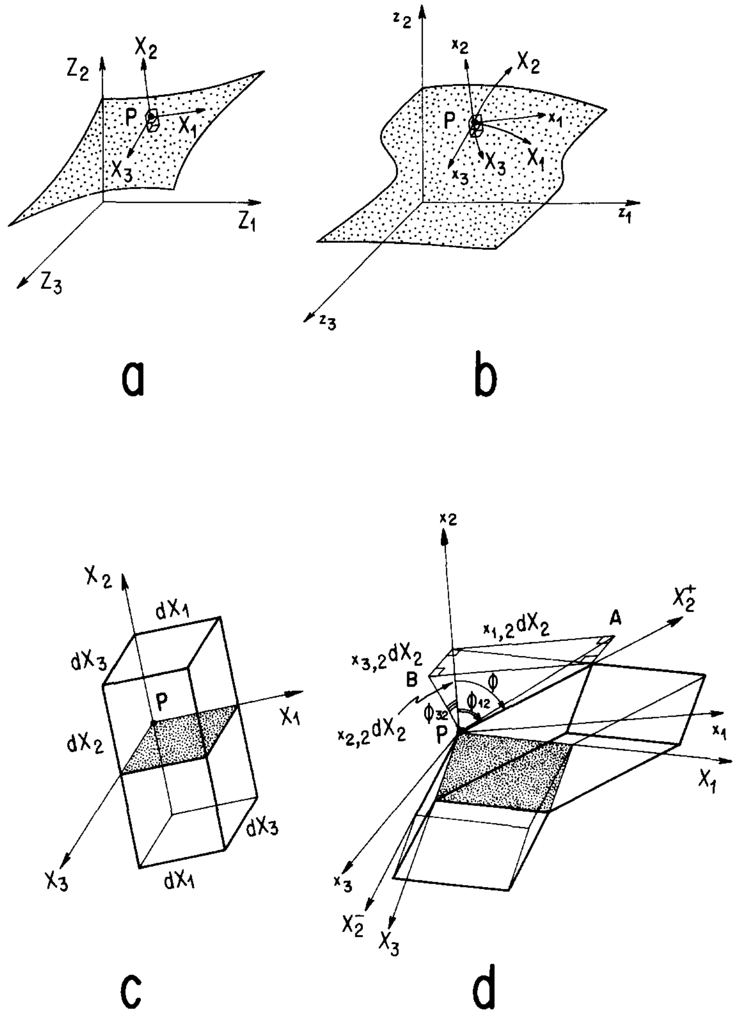

Fig. 1. Deformation at an interface (stippled). (a) Undeformed state, with common coordinate frame, $\boldsymbol{Z}$, and material frame $\boldsymbol{X}$ embedded in the interface such that X_2 is normal to the interface at point $\boldsymbol{P}$. (b) Deformed state, with common coordinate frame, z, deformed material frame $\boldsymbol{X}$, and new spatial frame $\boldsymbol{x}$ embedded in the interface, such that x_2 is normal to the interface at point $\boldsymbol{P}$. (c) Enlarged view of $\boldsymbol{P}$ in undeformed state (a), showing elementary cubes above and below interface (stippled). (d) Enlarged view of $\boldsymbol{P}$ in deformed state (b), showing parallelepipeds above and below interface, that result from deformation of cubes (c). Symbols are defined in text.

are rederived here, but in a simplified form which results from using rectangular Cartesian reference frames. Similar expressions, but in more general non-Cartesian form, have been used by Hobbs (1971).

After an initial introduction (chapter A), Truesdell & Toupin (1960) developed the field theory of kinematics of a continuous medium (chapter B), including sections on deformation and motion. This was followed by a discussion of singular surfaces (chapter C). As an interface mathematically is a special kind of singular surface, it is convenient to follow Truesdell & Toupin's approach.

Deformation of a continuous medium

To study this, imagine that a grid $\boldsymbol{X}$, is embedded in the undeformed material (Fig. 1a). After deformation (Fig. 1b), the original grid is distorted and one can describe this distortion with the aid of a new grid, $\boldsymbol{x}$. The two grids are compared using a common Cartesian frame, denoted $\boldsymbol{Z}$ or z according to whether the undeformed or deformed state is being described. For the analysis of an interface, it is convenient if $\boldsymbol{X}$ and $\boldsymbol{x}$ are also taken to be Cartesian, but each parallel to the interface at different times and thus oblique to the common frame and to one another. This simplifies the mathematics.

Following Truesdell & Toupin (1960, p. 326), the $\boldsymbol{X}$ are taken to be *material* coordinates (which deform with the material) and the $\boldsymbol{x}$, *spatial* coordinates (which are fixed and not deformable).

Deformation (Truesdell & Toupin 1960, p. 243) is understood to mean the mathematical transformation

$$\boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}), \tag{1}$$

whereas its inverse,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}), \tag{2}$$

is here termed the *reverse deformation*, for clarity.

An element of arc in the undeformed state, $\mathrm{d}\boldsymbol{X}$, undergoes a change in length and orientation to become an element of arc, $\mathrm{d}\boldsymbol{x}$, in the deformed state. Using Cartesian tensor notation, this is expressed as

$$\mathrm{d}x_i = x_{i,J}\mathrm{d}X_J;\ \ \mathrm{d}X_J = X_{J,m}\mathrm{d}x_m; \tag{3}$$

where $x_{i,J} = \partial x_i/\partial X_J$ is a tensor of *deformation gradients* (Truesdell & Toupin 1960, p. 245) and $X_{J,m} = \partial X_J/\partial x_m$ is the reciprocal tensor of *reverse deformation gradients*.

A squared element of arc in the undeformed state,

$$\mathrm{d}S^2 = (\mathrm{d}X_J)^2, \tag{4}$$

becomes in the deformed state, on substituting (3) into (4),

$$\mathrm{d}S^2 = c_{mp}\,\mathrm{d}x_m\,\mathrm{d}x_p; \tag{5}$$

where

$$c_{mp} = X_{J,m}X_{J,p} \tag{6}$$

is *Cauchy's tensor* (Truesdell & Toupin 1960, p. 257), which is symmetric ($c_{mp} = c_{pm}$). If $\mathrm{d}S^2$ is constant, then (4) describes an infinitesimal sphere in the undeformed state, and (5) describes the *strain ellipsoid* in the deformed state. The reciprocal of Cauchy's tensor is *Finger's tensor* (Truesdell & Toupin 1960, p. 263), defined as

$$c^{-1}_{mp} = x_{m,J}x_{p,J}. \tag{7}$$

Finger's tensor has the properties that (i) its proper numbers are the squares of the principal stretches, (ii) its proper vectors are the principal axes of the strain ellipsoid in the deformed state and (iii) it is expressed, by (7), in terms of the deformation gradients. See also Hobbs (1971) and De Paor (this issue).

Deformation at an interface

An interface, or internal boundary, is one example of the family of surfaces of discontinuity, or *singular surfaces* (Truesdell & Toupin 1960, p. 492). It is a *material surface* which is the common boundary of two regions, R^+ and R^-. If Ψ is a quantity which is continuous in each region and approaches definite limit values, Ψ^+ or Ψ^-, according to whether $\boldsymbol{x}$ approaches a point on the boundary from within one region or the other, then

$$[\Psi] \equiv \Psi^+ - \Psi^- \tag{8}$$

is the *jump* of Ψ across the interface. In general, the jump is a function of position along the interface.

Singular surfaces may be classified kinematically by regarding the deformation, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X})$, or its time history, the motion.

The *order* of a singular surface, with respect to $\boldsymbol{x}$, is defined as the order of the derivative of lowest order suffering a non-zero jump at the surface (Truesdell & Toupin, p. 517). In such a classification, *an interface is a first order material singularity*, because the deformation suffers no jump, but the deformation gradients may do so.

To illustrate this, choose for the undeformed state a Cartesian frame with origin at point $\boldsymbol{P}$, such that X_1 and X_3 lie tangent to the interface (Fig. 1a). For the deformed state, choose a new Cartesian frame with origin at the same material point $\boldsymbol{P}$, such that x_1 and x_3 lie tangent to the interface (Fig. 1b). But the lines X_1 and X_3 continue to lie tangent to the interface, as the latter is a material surface; therefore

$$x_{2,1} = X_{2,1} = x_{2,3} = X_{2,3} = 0. \tag{9}$$

Notice that the orientations chosen for X_1 and x_1 are arbitrary within the interface. Also the angle between X_1 and X_3 is no longer necessarily a right angle in the deformed state and the material line X_2 becomes oblique to the interface.

Now consider two adjoining elementary cubes in the undeformed state (Fig. 1c), one above and one below the interface, with edges along the reference frame and of lengths $\mathrm{d}X_I$. After a finite deformation (Fig. 1d), the cubes have become parallelepipeds, and the face they share, a parallelogram, lies within the interface; but the edge containing X_2 is no longer normal to the interface, the angle between x_2 and X_2 being ϕ. If the projections of ϕ onto the coordinate planes x_1x_2 and x_3x_2 are ϕ_{12} and ϕ_{32}, respectively, then the tangents of these angles are (Fig. 1d)

$$\begin{aligned} K_{12} &= \tan\phi_{12} = x_{1,2}/x_{2,2}, \\ K_{32} &= \tan\phi_{32} = x_{3,2}/x_{2,2}. \end{aligned} \tag{10}$$

Thus, ϕ_{12} is the *shear* (Truesdell & Toupin, p. 256) of the pair of lines x_1 and PA (the projection of X_2 onto the plane x_1x_2). Similarly ϕ_{32} is the shear of x_3 and PB (the projection of X_2 onto the plane x_3x_2). The tangents, K_{12} and K_{32}, are the *amounts* of these shears (Truesdell & Toupin, p. 293). Many authors would call K_{12} and K_{32} shear strains, but for large strains this nomenclature can lead to confusion with the shear components of the various strain tensors.

The amounts of shear, K_{12} and K_{32}, are measures of the state of strain at the interface. This can be shown using Finger's tensor. Because of the vanishing deformation gradients (9), the second column of c^{-1}_{mp} in (7) reduces to

$$c^{-1}_{12} = x_{1,2}x_{2,2};\ \ c^{-1}_{22} = x_{2,2}x_{2,2};\ \ c^{-1}_{32} = x_{3,2}x_{2,2}; \tag{11}$$

whence (10) becomes

$$K_{12} = c^{-1}_{12}/c^{-1}_{22};\ \ \ K_{32} = c^{-1}_{32}/c^{-1}_{22}. \tag{12}$$

Thus, it is a simple matter to calculate K_{12} and K_{32}, given the strain in the deformed state.

Of the non-vanishing deformation gradients, those suffering no jump at the interface are $x_{1,1}$, $x_{1,3}$, $x_{3,1}$ and $x_{3,3}$. These govern the state of strain within the interface, in other words, the shape of the stippled parallelogram in Fig. 1(d). For the analysis that follows, a convenient way of comparing values of a quantity Ψ across the interface is to use not the jump, but the *interface ratio*, defined here as

$$(\Psi) = \Psi^+/\Psi^-. \tag{13}$$

Thus, for the deformation gradients that suffer no jump,

$$(x_{1,1}) = (x_{1,3}) = (x_{3,1}) = (x_{3,3}) = 1. \tag{14}$$

The other non-vanishing deformation gradients, $x_{1,2}$, $x_{2,2}$ and $x_{3,2}$ may suffer jumps at the interface. The gradients $x_{1,2}$ and $x_{3,2}$ represent simple shears parallel to the interface, whereas $x_{2,2}$ represents a simple extension normal to the interface. Thus the deformation at an interface may be analysed in the same way as that in an ideal banded structure (Ramsay & Graham 1970, Cobbold 1977).

From (10), it is clear that K_{12} and K_{32} may both suffer jumps at the interface, as a result of a jump in $x_{2,2}$, or jumps in $x_{1,2}$ and $x_{3,2}$. To study how these jumps build up, it is useful to consider the time history of deformation, that is, the motion.

Motion of a continuous medium

The *motion* is the family of deformations,

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t), \ \boldsymbol{X} = \boldsymbol{X}(\boldsymbol{x}, t), \tag{15}$$

where t is the time. In their basic review, Truesdell & Toupin (1960, pp. 326–437) considered a fixed frame $\boldsymbol{x}$, which at time $t = 0$ becomes $\boldsymbol{X}$, so that

$$\boldsymbol{x} = f(\boldsymbol{z}), \ \boldsymbol{X} = f(\boldsymbol{Z}), \tag{16}$$

the function f being identical in both parts of equation (16). The coordinate scheme used in this paper is slightly different in that $\boldsymbol{x}$ and $\boldsymbol{X}$, although both Cartesian, are related to $\boldsymbol{z}$ and $\boldsymbol{Z}$ by different amounts of rigid rotation (Fig. 1). Nevertheless, because $\boldsymbol{x}$ and $\boldsymbol{X}$ are Cartesian, the basic results of Truesdell & Toupin are valid to within an overall rigid rotation (Truesdell & Toupin, p. 440), which of course does not modify the strains accumulated at the interface.

For a given material particle, $\boldsymbol{X}$, the *velocity*, $\dot{\boldsymbol{x}}$, is the rate of change of spatial position:

$$\dot{x}_k = \frac{\partial x_k}{\partial t}, \tag{17}$$

where $\boldsymbol{X}$ is held constant in time and space. From (17) and the first of (15),

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(\boldsymbol{X}, t), \tag{18}$$

so that velocity is a function of time for a given particle. However, $\boldsymbol{X}$ may be eliminated using the second of (15), giving

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}(\boldsymbol{x}, t), \tag{19}$$

where velocity is a function of time for a given place. A function such as (18), where $\boldsymbol{X}$ and t are the independent variables, is said to be in the *material description*; whereas a function such as (19), where $\boldsymbol{x}$ and t are the independent variables, is said to be in the *spatial description*.

Now consider the rate of change of a Cartesian tensor, $\boldsymbol{A}$, of order n. The *material derivative* of $\boldsymbol{A}$, $\dot{\boldsymbol{A}}$, is its rate of change relative to an observer situated upon a moving particle, so that $\boldsymbol{X}$ is constant. Thus for the material description, where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{X}, t)$,

$$\dot{A} = \left.\frac{\partial A(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}=\text{const}} = \frac{\mathrm{D}A}{\mathrm{D}t}, \tag{20}$$

whereas, for the spatial description, where $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x}, t)$,

$$\dot{A}\ldots = \left.\frac{\partial A \ldots (\boldsymbol{x}, t)}{\partial t}\right|_{\boldsymbol{x}=\text{const}} + A_{\ldots,q}\dot{x}_q. \tag{21}$$

The term on the far right of (21) is known as the *convection* of $\boldsymbol{A}$. Equation (20) is sufficient to find the material derivative of the deformation gradients:

$$\dot{\overline{x_{i,J}(\boldsymbol{X}, t)}} = \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\partial x_i}{\partial X_J}\right) = \frac{\partial \dot{x}_i}{\partial X_J} = \frac{\partial \dot{x}_i}{\partial x_m}\frac{\partial x_m}{\partial X_J} = \dot{x}_{i,m}x_{m,J}, \tag{22}$$

where the chain rule has been used. Equations (22) show that the rates of change of the deformation gradients depend upon the deformation gradients themselves and upon the *velocity gradients*, $\dot{x}_{i,m}$ (see also the discussion by Ramberg 1975). The latter can be expressed as the sum of a symmetric tensor, $\boldsymbol{d}$, and an antisymmetric one, $\boldsymbol{\omega}$:

$$\dot{x}_{i,m} = d_{im} + \omega_{im}. \tag{23}$$

As shown in detail by Truesdell & Toupin (1960, pp. 325–374), $\boldsymbol{d}$, the *stretching tensor*, represents a rate of pure strain, whereas $\boldsymbol{\omega}$, the *spin tensor*, represents a rate of rigid rotation.

Motion at an interface

Because the deformation suffers no jump at the interface, neither does the motion, nor its time-derivative, the velocity.

Of the velocity gradients, the following vanish because of the choice of spatial coordinate frame (x_2 always perpendicular to the interface):

$$\dot{x}_{2,1} = \dot{x}_{2,3} = 0. \tag{24}$$

Of the non-vanishing velocity gradients, $\dot{x}_{1,1}$, $\dot{x}_{1,3}$, $\dot{x}_{3,1}$ and $\dot{x}_{3,3}$ suffer no jumps, so that

$$(\dot{x}_{1,1}) = (\dot{x}_{1,3}) = (\dot{x}_{3,1}) = (\dot{x}_{3,3}) = 1. \tag{25}$$

The remaining velocity gradients, $\dot{x}_{1,2}$, $\dot{x}_{2,2}$ and $\dot{x}_{3,2}$, may suffer jumps across the interface.

Similarly, the material derivatives of the deformation gradients that may suffer jumps across the interface are $\dot{x}_{1,2}$, $\dot{x}_{2,2}$ and $\dot{x}_{3,2}$. Because of the vanishing velocity gradients (24), expansion of (22) with $J = 2$ yields

$$\begin{aligned} \dot{x}_{1,2} &= \dot{x}_{1,1}x_{1,2} + \dot{x}_{1,2}x_{2,2} + \dot{x}_{1,3}x_{3,2}, \\ \dot{x}_{2,2} &= \dot{x}_{2,2}x_{2,2}, \\ \dot{x}_{3,2} &= \dot{x}_{3,1}x_{1,2} + \dot{x}_{3,2}x_{2,2} + \dot{x}_{3,3}x_{3,2}. \end{aligned} \tag{26}$$

The material derivatives of K_{12} and K_{32} can be obtained by noticing that in (10) they are functions of the deformation gradients and therefore of $\boldsymbol{X}$ and t. Using (20), the quotient rule for partial differentiation and (22),

$$\begin{aligned} \dot{K}_{12} = {} & (\dot{x}_{1,1}x_{1,2} + \dot{x}_{1,2}x_{2,2} + \dot{x}_{1,3}x_{3,2})/x_{2,2} \\ & - (\dot{x}_{2,1}x_{1,2} + \dot{x}_{2,2}x_{2,2} + \dot{x}_{2,3}x_{3,2})x_{1,2}/(x_{2,2})^2. \end{aligned} \tag{27}$$

Simplifying and using (10) and (24),

$$\dot{K}_{12} = K_{12}(\dot{x}_{1,1} - \dot{x}_{2,2}) + \dot{x}_{1,2} + K_{32}\dot{x}_{1,3}. \quad (28)$$

Similarly,

$$\dot{K}_{32} = K_{32}(\dot{x}_{3,3} - \dot{x}_{2,2}) + \dot{x}_{3,2} + K_{12}\dot{x}_{3,1}. \quad (29)$$

In equations (28) and (29), $\dot{x}_{1,2}$ and $\dot{x}_{3,2}$ are the *amounts of simple shearing*, or rates of simple shear, with shearing planes parallel to the interface and shear directions parallel to x_1 and x_3; whereas $\dot{K}_{12}$ and $\dot{K}_{32}$ are rates of change of total amounts of shear.

If the motion is confined to the x_1x_2 plane, (28) reduces to

$$\dot{K}_{12} = K_{12}(\dot{x}_{1,1} - \dot{x}_{2,2}) + \dot{x}_{1,2}, \quad (30)$$

which expression has been derived before (Bayly 1964, Cobbold 1976, Bayly & Cobbold 1979), although not as rigorously as in the present treatment, where it is shown to be exact. The geometrical significance of (30) can be appreciated by considering two limits.

(i) If $\dot{x}_{1,1} = \dot{x}_{2,2}$, the motion is a simple shearing, $\dot{x}_{1,2}$, plus a uniform dilatation (rate of dilation). Equation (30) reduces to $\dot{K}_{12} = \dot{x}_{1,2}$, so that K_{12} changes by simple shearing alone. A uniform dilatation has no effect on angles.

(ii) If $\dot{x}_{1,2} = 0$, (30) reduces to

$$\dot{K}_{12}/K_{12} = \dot{x}_{1,1} - \dot{x}_{2,2} = \dot{\lambda}_1/\lambda_1 - \dot{\lambda}_2/\lambda_2, \quad (31)$$

where λ is the stretch, or final length divided by original length (Truesdell and Toupin 1960, p. 255). Integration of (31) yields

$$K_{12}/K_{12}|_0 = \lambda_1/\lambda_2, \quad (32)$$

where $K_{12}|_0$ is a value of K_{12} before the stretches, λ_1 and λ_2, are imposed. Equation (32) is a well-known relationship for change in angle due to plane strain (see Ramsay 1967, p. 67). Thus, (31) shows how K_{12} is sensitive to the differential simple stretching, $\dot{x}_{1,1} - \dot{x}_{2,2}$. In general, the effects of simple shearing and simple stretching are additive (equation 29) because they are rates, that is, derivatives of the total deformation.

In three dimensions, the general equations (28) and (29) are almost of the form (30) and valid each for one coordinate plane; but there are additional terms, $K_{32}\dot{x}_{1,3}$ and $K_{12}\dot{x}_{3,1}$, which result from shearing within the plane of the interface.

Now consider what conditions are necessary or sufficient to ensure that

$$(K_{12}) = (K_{32}) = R, \quad (33)$$

where R is a constant throughout time, but not space; thus the problem is to find conditions such that the interface ratios of K_{12} and K_{32} remain constant and both equal to R. From the definitions (10) and from (33), sufficient (but not necessary) conditions are

$$(x_{1,2}) = (x_{3,2}) = R; \; (x_{2,2}) = 1. \quad (34)$$

On writing out each of (34) in full and taking the material derivatives (Appendix),

$$(\dot{x}_{1,2}) = (\dot{x}_{3,2}) = R; \; (\dot{x}_{2,2}) = 1. \quad (35)$$

Necessary and sufficient conditions upon the velocity gradients, for (34) and (35) to hold, are derived in the Appendix. They are

$$(\dot{x}_{1,2}) = (\dot{x}_{3,2}) = R; \; (\dot{x}_{2,2}) = 1. \quad (36)$$

From the above argument, conditions (36) are sufficient, but not strictly necessary, to ensure (33). Thus, for the interface ratios of finite amounts of shear, (K_{12}) and (K_{32}), to have a constant value R throughout the motion, it is sufficient that (i) the ratios $(\dot{x}_{1,2})$ and $(\dot{x}_{3,2})$ of simple shearings parallel to the interface retain a constant value R and that (ii) the stretching, $\dot{x}_{2,2}$, normal to the interface, suffer no jump.

From the definition of the spin, $\boldsymbol{\omega}$, as an antisymmetric tensor, it is necessary that $\omega_{11} = \omega_{22} = \omega_{33} = 0$. Also, from (23), (24) and the symmetry of d_{im}, $\omega_{21} = -\omega_{12} = -d_{12}$ and $\omega_{32} = -\omega_{32} = -d_{32}$. Therefore,

$$\begin{gathered}\dot{x}_{1,1} = d_{11},\ \dot{x}_{1,2} = 2d_{12},\ \dot{x}_{1,3} = d_{13} + \omega_{13},\\ \dot{x}_{2,1} = 0,\ \dot{x}_{2,2} = d_{22},\ \dot{x}_{2,3} = 0,\\ \dot{x}_{3,1} = d_{13} - \omega_{13},\ \dot{x}_{3,2} = 2d_{32},\ \dot{x}_{3,3} = d_{33}.\end{gathered} \quad (37)$$

Substitution of (37) in (36) yields

$$(d_{12}) = (d_{32}) = R; \; (d_{22}) = 1. \quad (38)$$

The dilatation (rate of dilation) is

$$d_{kk} = d_{11} + d_{22} + d_{33}. \quad (39)$$

As $\dot{x}_{1,1} = d_{11}$ and $\dot{x}_{3,3} = d_{33}$ suffer no jumps at the interface (equations 25), the second of (38) implies that the dilatation suffers no jump; in other words

$$(d_{kk}) = 1. \quad (40)$$

Before examining the mechanical conditions under which (38) may hold, it is worth exploring some geometrical implications of (33).

First, if $x_{2,2}{}^+ = x_{2,2}{}^-$ and $(x_{1,2}) = (x_{3,2}) = R$, the axes x_2, X_2^+ and X_2^- all lie in one plane (Fig. 1d). Thus, if the spatial frame is rotated about x_2, until X_2^+ and X_2^- lie in the plane x_1x_2, then $x_{3,2}^+ = x_{3,2}^- = 0$ and (K_{12}) measures the refraction of X_2. Otherwise, if the spatial frame is rotated about x_2 until x_3 coincides with X_3, then (K_{12}) measures the dihedral refraction of the plane X_3X_2. Finally, if the spatial frame is rotated until x_1 coincides with X_1, (K_{32}) measures the dihedral refraction of the plane X_1X_2. Thus, R may be obtained by measuring the refraction of any line or plane that was normal to the interface before deformation.

MECHANICS

Following Truesdell & Toupin (1960, p. 543), the stress tensor is denoted $\boldsymbol{t}$. To preserve the balance of forces at a coherent interface, the following stress components suffer no jump:

$$(t_{12}) = (t_{22}) = (t_{32}) = 1. \quad (41)$$

This completes the analysis of stress. The next step is to consider rheology.

So as to satisfy conditions (40), only incompressible materials will be considered. As the spatial axes x_1 and x_3 are chosen arbitrarily within the interface, without reference to any specific material lines, the material must have *at least* a uniaxial symmetry of rheological properties (transverse isotropy) about the interface normal, x_2. This class of symmetry includes spherical symmetry (full isotropy) as a special sub-class.

In the simple rheological models that follow, the material is assumed to be incompressible and fully isotropic.

Incompressible neo-Hookean elastic solid

In the incompressible neo-Hookean model (Truesdell & Noll 1965, p. 350) the stress $\boldsymbol{t}$ is directly proportional to the finite strain, which is elastic and fully recoverable:

$$ {}_0t_{ij} = G\,{}_0c^{-1}_{ij};\ \mathrm{III}_{{}_0c^{-1}} = 1; \tag{42}$$

where

$$ {}_0t_{ij} = t_{ij} - \delta_{ij}t_{kk} \tag{43}$$

is the deviatoric stress tensor, G is a constant (the rigidity modulus) and

$$ {}_0c^{-1}_{ij} = c^{-1}_{ij} - \delta_{ij}c^{-1}_{kk} \tag{44}$$

is the deviatoric part of Finger's tensor, defined in (7). The third invariant, $\mathrm{III}_{{}_0c^{-1}}$, is equal to unity in (42) because there is no volume change.

Expansion of (41) yields, for the shear components,

$$ {}_0t_{12} = t_{12} = Gc^{-1}_{12};\ {}_0t_{32} = t_{32} = Gc^{-1}_{32}. \tag{45}$$

Substitution of (45) into (41) gives

$$ (c^{-1}_{12}) = (c^{-1}_{32}) = (1/G). \tag{46}$$

As the material is incompressible and the area change within the interface suffers no jump, then

$$ (x_{2,2}) = 1. \tag{47}$$

Substitution of (46) and (47) into (11) gives

$$ (x_{1,2}) = (x_{3,2}) = (1/G);\ (x_{2,2}) = 1, \tag{48}$$

which are of the form (34) and sufficient to ensure that

$$ (K_{12}) = (K_{32}) = (1/G). \tag{49}$$

Thus, in an incompressible neo-Hookean model, the ratios of finite amounts of shear are at all times equal to the inverse rigidity ratio. This very simple result provides one with a powerful method for determining rigidity ratios in real physical models made from neo-Hookean rubberlike materials. For real rocks, where recoverable elastic strains of more than 4% or so have never been reported, the result (49) is not likely to be useful.

Newtonian fluid

A Newtonian fluid is by definition isotropic and incompressible. The rheological behaviour is completely specified by the equations of state (flow law)

$$ {}_0t_{ij} = 2\mu d_{ij};\ d_{kk} = 0; \tag{50}$$

where μ is a constant scalar, the *shear viscosity*. For shear components, (50) gives

$$ {}_0t_{12} = t_{12} = 2\mu d_{12};\ {}_0t_{32} = t_{32} = 2\mu d_{32}. \tag{51}$$

Substitution of (51) into (41), with $d_{kk} = 0$ in (39) yields (see also Treagus 1973, 1981)

$$ (d_{12}) = (d_{32}) = (1/\mu);\ (d_{22}) = 1, \tag{52}$$

which is of the form (38) because μ is constant.

Thus Newtonian rheology on both sides of the interface is sufficient to ensure that the finite ratios (K_{12}) and (K_{32}) remain constant; conversely, measurement of these ratios yields the inverse viscosity ratio, $(1/\mu)$. Current work with H. Hugon shows that this simple result provides a powerful method for measuring or checking viscosity ratios in real physical models. Furthermore, the long-term creep of rocks may be approximated by Newtonian behaviour, and the method may be applicable to determining viscosity ratios at natural interfaces. It also provides a basis for predicting strain patterns at interfaces (Treagus, this issue).

Reiner–Rivlin fluid

A Reiner–Rivlin fluid is incompressible and isotropic with flow-law (Truesdell & Noll 1965, Hobbs 1972, Ferguson 1979)

$$ {}_0t_{ij} = 2\mu d_{ij};\ d_{kk} = 0; \tag{53}$$

where μ, the *shear viscosity*, is not constant, but a function of the second or third principal invariants of the stretching tensor:

$$ \begin{aligned} \mathrm{II}_{\boldsymbol{d}} &= d_1d_2 + d_2d_3 + d_3d_1 = -\tfrac{1}{2}(d_1^2 + d_2^2 + d_3^2),\\ \mathrm{III}_{\boldsymbol{d}} &= d_1d_2d_3. \end{aligned} \tag{54}$$

Therefore, μ is also, in general, a function of position within the deforming fluid.

A fairly wide class of behaviours may be obtained by taking μ as a function only of the second invariant, $\mathrm{II}_{\boldsymbol{d}}$. Thus, define an octahedral shear stress, τ, and an octahedral shearing, d:

$$ \begin{aligned} \tau^2 &= \tfrac{1}{2}\,{}_0t_{ij}\,{}_0t_{ij} = -\mathrm{II}_{{}_0t},\\ d^2 &= \tfrac{1}{2}d_{ij}d_{ij} = -\mathrm{II}_{\boldsymbol{d}}. \end{aligned} \tag{55}$$

Expansion of the second equation gives

$$ 2d^2 = d_{11}^2 + d_{22}^2 + d_{33}^2 + 2d_{12}^2 + 2d_{32}^2 + 2d_{31}^2. \tag{56}$$

Substitution of (53) into (55) yields the simple equation

$$ \tau = 2\mu d, \tag{57}$$

where μ is a function of d, or of τ (Nye 1953).

At an interface between two Reiner–Rivlin fluids, from (53) and (41),

$$ (d_{12}) = (d_{32}) = (1/\mu);\ (d_{22}) = 1. \tag{58}$$

If μ is taken as a function of d, then from (56) it is clear that $(1/\mu)$ is not generally constant. Thus, conditions (38) are not satisfied and the viscosity ratio cannot in general be obtained by measurement of the finite ratio of amounts of shear.

To see this more clearly, consider a well-known special case of the Reiner–Rivlin fluid, that is the power-law fluid, where

$$\mu = d^{(1/n)-1}/2A^{1/n}, \tag{59}$$

so that, from (58),

$$d = A\tau^n, \tag{60}$$

n being a constant, the stress exponent, and A being another constant. This is the Weertman (1968) equation, observed in simple laboratory experiments on polycrystalline rock, under conditions where the deformation mechanism is dislocation creep and the stretch history is constant. Although the use of this model has been criticized for more complex stretch histories (Ferguson 1979), it is so popular at the moment that its effects on interfaces are worth examining.

From (59) and (56), it is clear that in general $(1/\mu)$ is not constant; but it does have two limiting constant values, for two special classes of motion. Thus, if $(n) = 1$ and

$$d_{12}/d = d_{32}/d = 0, \tag{61}$$

the motion includes no shearing upon shearing planes parallel to the interface. From (61), (56) and (59), $(d) = 1$ and

$$(1/\mu) = (A)^{1/n}. \tag{62}$$

Thus, the inverse viscosity ratio locally has a constant value if the motion locally includes no shearing along the interface. Under these conditions, K_{12} and K_{32} never depart from zero; but if the motion tends towards this limit, the finite ratios of K_{12} and K_{32} will tend towards the limiting value given in (62).

The second limiting constant value of the inverse viscosity ratio occurs if

$$d_{11}/d = d_{22}/d = d_{33}/d = d_{31}/d = 0. \tag{63}$$

This means that the motion is locally a simple shearing parallel to the interface within which there is no surface straining. From (63), (56), (58) and (59),

$$(1/\mu) = (A). \tag{64}$$

Thus for simple shearing the finite ratios of K_{12} and K_{32} will locally have the value (A).

In general, the motion at an interface includes components of both simple shearing and surface straining, so that the viscosity ratio is not necessarily constant in time or space. If there is no jump in n across the interface, the inverse viscosity ratio is in the range between (A) and $(A)^{1/n}$. By estimating these limiting values, one can estimate n. If $n = 1$, the fluid is Newtonian and (62) and (64) yield a unique viscosity ratio, which is constant. If n is very large, the rheological behaviour approximates that of an ideal rigid-plastic model. The viscosity ratio tends to unity (equation 62) if there is little shearing parallel to the interface: under these conditions the interface no longer has any mechanical or kinematic significance. In contrast, if shearing predominates, the inverse viscosity ratio (equation 64) tends to a limiting value (A). Finally, for moderate values of n, the viscosity ratio is closest to unity at points where there is least shearing along the interface: the non-linearity of the power-law inhibits the development of strain discontinuities at these points.

CONCLUSIONS

(1) At a coherent interface there may be jumps in the amount of extension normal to the interface and the amount of shear parallel to it; that is, discontinuities in total strain and rigid rotation.

(2) At a given instant of time, there may be jumps in the stretching normal to the interface and the shearing parallel to it.

(3) The ratio (K) of total amounts of shear above and below an interface is equal to the ratio of amounts of simple shearings, provided the latter ratio is invariant with time and that there are no volume changes.

(4) In neo-Hookean elastic solids, the finite ratio (K) is equal to the inverse rigidity ratio at the interface.

(5) In Newtonian fluids, the inverse viscosity ratio is equal to the ratio of amounts of simple shearings and the viscosity ratio can therefore be uniquely determined by measurement of (K) alone.

(6) The ratio of shearings is not necessarily invariant with time in Reiner–Rivlin fluids, so that the viscosity ratio is not uniquely determined by measurement of (K) alone.

(7) In power-law fluids, the viscosity ratio is constant for two special classes of motion, one with no simple shearing parallel to the interface, the other with simple shearing alone. The ratio of material constants, (A), is uniquely determined by measurements of (K) at a site known to have undergone simple shearing alone; whereas the stress-exponent, n, if constant across the interface, is uniquely determined by further measuring (K) at a site known to have undergone almost no simple shearing.

(8) The above theoretical results are applicable to measurement of rheological contrast in nature or in experimental models, where there are passive markers known to have been normal to an interface before deformation, or where there is any other information allowing one to calculate the amounts of shear parallel to the interface. In this sense, an interface is an inbuilt rheometer.

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APPENDIX: NECESSARY AND SUFFICIENT CONDITIONS FOR $(x_{1,2}) = (x_{3,2}) = R$; $(x_{2,2}) = 1$

The problem is to find necessary and sufficient conditions upon the velocity gradients, $\dot{x}_{i,j}$, such that the deformation gradients satisfy at all times

$$(x_{1,2}) = (x_{3,2}) = R;\ (x_{2,2}) = 1, \tag{A1}$$

where R is a constant.

Written out in full, equations (A1) are

$$x_{1,2}{}^{+} = Rx_{1,2}{}^{-};\ x_{3,2}{}^{+} = Rx_{3,2}{}^{-};\ x_{2,2}{}^{+} = x_{2,2}{}^{-}. \tag{A2}$$

The material derivatives of (A2) are

$$\dot{x}_{1,2}{}^{+} = R\dot{x}_{1,2}{}^{-};\ \dot{x}_{3,2}{}^{+} = R\dot{x}_{3,2}{}^{-};\ \dot{x}_{2,2}{}^{+} = \dot{x}_{2,2}{}^{-}; \tag{A3}$$

whence (equation 35)

$$(\dot{x}_{1,2}) = (\dot{x}_{3,2}) = R;\ (\dot{x}_{2,2}) = 1. \tag{A4}$$

Now the material derivatives in (A4) can be expressed in terms of the deformation gradients and velocity gradients (equation 26):

$$\begin{aligned} \dot{x}_{1,2} &= \dot{x}_{1,1}x_{1,2} + \dot{x}_{1,2}x_{2,2} + \dot{x}_{1,3}x_{3,2}, \\ \dot{x}_{2,2} &= \dot{x}_{2,2}x_{2,2}, \\ \dot{x}_{3,2} &= \dot{x}_{3,1}x_{1,2} + \dot{x}_{3,2}x_{2,2} + \dot{x}_{3,3}x_{3,2}. \end{aligned} \tag{A5}$$

Substituting the second of (A1) and the second of (A4) into the second of (A5),

$$(\dot{x}_{2,2}) = 1. \tag{A6}$$

Thus, (A6) is a necessary condition to ensure the second of (A1). Furthermore it is sufficient, for substitution of (A6) into the second of (A5) yields

$$(\dot{x}_{2,2}) = (x_{2,2}). \tag{A7}$$

The solution of (A7) is

$$(x_{2,2}) = \text{constant}. \tag{A8}$$

With the conditions that at time $t = 0$, $x_{2,2} = 1$, (A8) becomes the second of (A1) and sufficiency is proved.

The first and third of equations (A5), written for both sides of the interface, are

$$\begin{aligned} \dot{x}_{1,2}{}^{+} &= \dot{x}_{1,1}x_{1,2}{}^{+} + \dot{x}_{1,2}{}^{+}x_{2,2} + \dot{x}_{1,3}x_{3,2}{}^{+}, \\ \dot{x}_{1,2}{}^{-} &= \dot{x}_{1,1}x_{1,2}{}^{-} + \dot{x}_{1,2}{}^{-}x_{2,2} + \dot{x}_{1,3}x_{3,2}{}^{-}, \\ \dot{x}_{3,2}{}^{+} &= \dot{x}_{3,1}x_{1,2}{}^{+} + \dot{x}_{3,2}{}^{+}x_{2,2} + \dot{x}_{3,3}x_{3,2}{}^{+}, \\ \dot{x}_{3,2}{}^{-} &= \dot{x}_{3,1}x_{1,2}{}^{-} + \dot{x}_{3,2}{}^{-}x_{2,2} + \dot{x}_{3,3}x_{3,2}{}^{-}. \end{aligned} \tag{A9}$$

Substituting the first two of (A3) into the first and third of (A9) and multiplying the second and fourth of (A9) by R,

$$\begin{aligned} R\dot{x}_{1,2}{}^{-} &= R\dot{x}_{1,1}x_{1,2}{}^{-} + \dot{x}_{1,2}{}^{+}x_{2,2} + R\dot{x}_{1,3}x_{3,2}{}^{-}, \\ R\dot{x}_{1,2}{}^{-} &= R\dot{x}_{1,1}x_{1,2}{}^{-} + R\dot{x}_{1,2}{}^{-}x_{2,2} + R\dot{x}_{1,3}x_{3,2}{}^{-}, \\ R\dot{x}_{3,2}{}^{-} &= R\dot{x}_{3,1}x_{1,2}{}^{-} + \dot{x}_{3,2}{}^{+}x_{2,2} + R\dot{x}_{3,3}x_{3,2}{}^{-}, \\ R\dot{x}_{3,2}{}^{-} &= R\dot{x}_{3,1}x_{1,2}{}^{-} + R\dot{x}_{3,2}{}^{-}x_{2,2} + R\dot{x}_{3,3}x_{3,2}{}^{-}. \end{aligned} \tag{A10}$$

Comparing equations (A10) in pairs,

$$\dot{x}_{1,2}{}^{+} = R\dot{x}_{1,2}{}^{-};\ \dot{x}_{3,2}{}^{+} = R\dot{x}_{3,2}{}^{-}; \tag{A11}$$

whence

$$(\dot{x}_{1,2}) = (\dot{x}_{3,2}) = R. \tag{A12}$$

These are therefore necessary conditions to ensure the first of equations (A1). Sufficiency may be demonstrated using a recurrence method.

Assume first of all that (A1) holds at a given moment of time, t_m. Then substitute (A1) and (A12) into (A9). Corresponding pairs of terms on the right hand side of each pair of equations (A9) are in the ratio R; therefore (A4) holds at time t_m. By analogy with equations (A7) and (A8), it follows that (A1) must hold at all $t > t_m$. In other words, should (A1) hold once, it will hold for ever after. Now consider what happens at time $t = 0$. By definition, $x_{1,2} = x_{3,2} = 0$ at the onset of deformation, so that $(x_{1,2})$ and $(x_{3,2})$ are indeterminate; but substituting the null values into (A9), it yields (A4). Therefore, (A4) is also valid at time $t = 0$. Integrating over a small time increment, it is clear that the first (infinitesimal) values of $x_{1,2}$ and $x_{3,2}$ will have interface ratios equal to R, in other words (A1) holds. But it has already been shown that if (A1) holds once, it will hold for ever after. Therefore, it has been shown that (A12) and (A6) are sufficient conditions to ensure (A1). Earlier, it was shown that they are necessary.

A more formal proof of sufficiency is available if motion is confined to the x_1x_2 plane. Equations (A9) reduce to

$$\dot{x}_{1,2} = \dot{x}_{1,1}x_{1,2} + \dot{x}_{1,2}x_{2,2}. \tag{A13}$$

With $x_{1,2} = u(t)$, $\dot{x}_{1,1} = f(t)$, $\dot{x}_{1,2}x_{2,2} = g(t)$, (A13) becomes

$$\frac{\mathrm{D}u}{\mathrm{D}t} - uf(t) = g(t), \tag{A14}$$

which is a general linear first-order differential equation in u. Using the integrating factor,

$$Q(t) = \exp \int -f(t)\,\mathrm{d}t = \exp \int -\dot{x}_{1,1}\,\mathrm{d}t, \tag{A15}$$

the general solution of (A14) can be written

$$Qu = \int Qg(t)\,\mathrm{d}t + b, \tag{A16}$$

where b is a constant. At time $t = 0$, $Q = 0$ and therefore $b = 0$. Hence

$$Qx_{1,2} = \int_0^t Q\dot{x}_{1,2}x_{2,2}\,\mathrm{d}t. \tag{A17}$$

But from (A15), $\langle Q \rangle = 1$. Therefore, from (A17), (A8) and (A12),

$$\langle x_{1,2} \rangle = \left\langle \int_0^t Q \dot{x}_{1,2} x_{2,2} \, \mathrm{d}t \right\rangle = R, \tag{A18}$$

where R, being a constant, has been taken out of one of the integrals. The first and third of equations (A5) are a system of 2 general linear first-order differential equations in $x_{1,2}$ and $x_{3,2}$. Apparently, no standard method is known for finding a solution to such a system in terms of elementary functions, if such a solution exists at all. Therefore, no formal proof of sufficiency is offered for the three-dimensional equations (A5). Nevertheless the recurrence method described earlier appears to be rigorous enough. It may also be applied directly to equations (28) and (29), in other words to show that (A6) and (A12) are sufficient (but not necessary) conditions to ensure that the finite amounts of shear, K_{12} and K_{32}, have constant interface ratios equal to R.